1.1 For all periods, play Cc if history is Cc for all prior periods. If not, play Dd. Payoffs for cooperating on the equilibrium path are $\frac{2}{1-\delta}$ and deviating yields $5 + \frac{\delta}{1-\delta}$. Then cooperating is optimal for

$$\frac{2}{1-\delta} > 5 + \frac{\delta}{1-\delta}$$

$$\frac{2-\delta}{1-\delta} > 5$$

$$2 - \delta > 5(1 - \delta)$$

$$4\delta > 3$$

$$\delta > 3/4$$

Off the equilibrium path, the payoff is $\frac{1}{1-\delta}$ for cooperating and $\frac{\delta}{1-\delta}$ for deviating, so it is never optimal to deviate here. Thus, this is a SPNE.

1.2

For periods $1 + 2 \cdot k \in \mathbb{N}$, Player 1 plays C, player 2 plays c if history is Cc for all periods $1 + 2 \cdot k$ and Dd for all periods $2 \cdot k$, else player 2 plays d.

For periods $2 \cdot k \in \mathbb{N}$, Player 1 plays D, player 2 plays d if history is Cc for all periods $1 + 2 \cdot k$ and Dd for all periods $2 \cdot k$, else player 1 plays D.

The best time to deviate will be at the beginning of a run of cooperate, when you have the most to lose by sticking to the strategy. Let’s consider player 1, who has to cooperate more often and thus has a greater incentive to deviate. Given $\delta$, the payoff for continuing is at the beginning
of player 1’s cooperation is \[\sum_{t=0}^{\infty} \delta^{2t} 2 + \delta \sum_{t=0}^{\infty} \delta^{2t} = \frac{2}{1-\delta^2} + \frac{\delta}{1-\delta^2} = \frac{2+\delta}{1-\delta^2} \]. The payoff for deviating is \(5 + \frac{\delta}{1-\delta}\). Then cooperating is optimal for \(\frac{2+\delta}{1-\delta^2} > 5 + \frac{\delta}{1-\delta}\)

\[\delta > \frac{\sqrt{3}}{2}\]

1.3

For periods \(1 + 50 \cdot k\) to \(21 + 50 \cdot k\), \(k \in \mathbb{N}\), Player 1 plays D, player 2 plays c if history is Dc for all periods \(1 + 50 \cdot k\) to \(21 + 50 \cdot k\) and Cd for all periods \(22 + 50 \cdot k\) to \(50 + 50 \cdot k\), else player 2 plays d.

For periods \(22 + 50 \cdot k\) to \(50 + 50 \cdot k\), \(k \in \mathbb{N}\), Player 1 plays C, player 2 plays d if history is Dc for all periods \(1 + 50 \cdot k\) to \(21 + 50 \cdot k\) and Cd for all periods \(22 + 50 \cdot k\) to \(50 + 50 \cdot k\), else player 1 plays D.

The best time to deviate will be at the beginning of a run of cooperate, when you have the most to lose by sticking to the strategy. Let’s consider player 1, who has to cooperate more often and thus has a greater incentive to deviate. Given \(\delta\), the payoff for continuing is at the beginning of player 1’s cooperation is \[\sum_{t=29}^{50} \delta^t 5 + \sum_{t=79}^{100} \delta^t 5 + \ldots\] This is less than \[\sum_{t=29}^{50} \delta^{50} 5 + \sum_{t=79}^{100} \delta^{100} 5 + \ldots\]. Define \(\delta^* = \delta^{50}\). Then we have \[\sum_{t=1}^{\infty} 105\delta^t 5 = 105\delta^{50} / (1 - \delta^{50})\]. Deviating to D, we have Dd forever, a payoff of \[\sum_{t=0}^{\infty} \delta^t = 1 / (1 - \delta)\]. For \(\delta\) sufficiently large, the continuation payoff is approximately \(105 / (1 - \delta^{50})\), which is greater than \(105/50 / (1 - \delta)\), which is greater than \(1 / (1 - \delta)\), so there is no profitable deviation.

1.4

Internal consistency violated for all of them, as the punishment phase has a lower NPV for both players than any history on the equilibrium path. Thus the agents would prefer to return to a cooperation history after a defection.

2.1

The ex post efficient provision rule is to provide the good whenever \(\sum_i v_i \geq C\). PO allocations: All allocations \(a_i\) st \(\sum_i a_i = \sum_i v_i - C\) (and \(a_i \geq 0\) with or without this proviso is an OK reading of the problem).
2.2
 Dominant strategy eqm: everyone reports true type. Payoff only affected by report if it changes the provision decision, and good provided iff $\pi_i = v_i - t_i = v_i + \sum_{i \neq j} \hat{v}_j - C \geq 0$.

2.3
$\sum_i t_i - C = \sum_i (C - \sum_{i \neq j} \hat{v}_j) - C = (n - 1)(C - \sum_i v_i)$

2.4
NE: Any vote where the provision wins or loses by more than 1 vote (no one is decisive). Any vote where the provision wins or loses by less than 2 votes, and everyone votes for their preferred outcome. Dominant strategy Eqm: Vote your true preference, since voting only changes payoffs via provision outcome. if $v_i - C/n > 0$, you strictly prefer implementation.

2.5
Not generally. Suppose there are 9 agents, $C=90$, and 5 agents have valuation $v_i = 9$, while 4 agents have valuation $v_i = 100$. Then the total payoff for implementation is $445 - 90 = 355 > 0$. However, every agent will pay $C/9 = 10$ if the policy is implemented, so the $v = 9$ agents get negative utility from provision and will vote against it. Because we don’t have transferable utility, we end up with a suboptimal allocation.

3.1
$x \succ y$
$y \succ z$
$z \succ x$
We have a cycle, so we cannot rank preferences from least to greatest as we’ve violated transitivity.

3.2.1
Well defined. Every choice gets a natural number associated with it, and preference and indifference are determined by comparing the natural numbers, so since weak inequalities are complete in the natural numbers, this SWF is complete. It is also transitive, since inequality is transitive on the natural numbers.
True. Suppose not: Then there is some alternative w that is weakly preferred by all agents to the alternative selected by the borda count, v, and strictly preferred by at least one. But then w gets at least 1 more point in the borda count than v, since no agent strictly prefers v to w and at least one agent strictly prefers w to v, so they must either prefer w to 1 option while v is preferred to none or prefer w to both options while v is preferred to just 1 or 0. But then w has a higher borda count and is the winner. Contradiction.

3.2.3
True. Consider the preferences

\[
\begin{array}{ccc}
  j & k & i \\
  x & x & z \\
  y & y & x \\
  z & z & y \\
\end{array}
\]
zPix, but xPz.

3.2.3
True. Consider the preferences

\[
\begin{array}{ccc}
  j & k & i \\
  x & x & z \\
  y & y & x \\
  z & z & y \\
\end{array}
\]
zPix, but xPz.

3.2.4
The borda count does not satisfy IIA

\[
\begin{array}{ccc}
  1 & 2 & 3 \\
  x & x & z \\
  y & y & x \\
  z & z & y \\
  1 & 2 & 3 \\
  y & y & z \\
  x & x & x \\
  z & z & y \\
\end{array}
\]
In both P and P’, the first two voters prefer y to z and the third voter prefers z to y. However, in the first case y and z both get 2 points, thus yPz and zPy. In the second case, y gets 4 points
and \( z \) gets 2, so \( yP'z \), but not \( zP'y \). Thus we have \( zPy \) and not \( zP'y \), despite all agents in each agreeing on the ordering of \( z \) and \( y \) with their corresponding counterpart. So IIA is violated.

4.1

Let’s solve this problem as a dynamic program, where the value of playing as player \( i \) given that player \( j \) is proposing is \( V_{ij} \), which in equilibrium will be the offered payoff for player \( i \) when player \( j \) is proposing. Define Alice, Bob, and Curt as \( A \), \( B \), and \( C \). Define \( \pi_C = 1 - \pi_A - \pi_B \).

Notice that \( V_{AB} \geq \delta(V_{AA}/3 + V_{AB}/3 + V_{AC}/3) \), since Alice can always wait a period and become the receiver again with probability \( 2/3 \), else become the sender. In fact, we know that Bob will offer the absolute minimum possible to get players 2 and 3 to accept as this is a zero sum game, so this will be an equality:

\[
V_{AB} = \delta(V_{AA}/3 + V_{AB}/3 + V_{AC}/3)
\]

Also note that Alice has the same outside option when Curt proposes, so \( V_{AC} = V_{AB} \), and more generally players have the same payoffs for being receivers regardless of the proposer. Define \( V_{i,-i} = W_i \), \( V_{ii} = V_i \). Then we can simplify the condition to

\[
W_A = \delta(V_A/3 + 2/3 * W_A)
\]

\[
W_A = \frac{\delta/3}{3/3 - 2\delta/3} V_A = \frac{\delta}{3 - 2\delta} V_A
\]

Similarly, we have

\[
W_B = \frac{\delta}{3 - 2\delta} V_B
\]

\[
W_C = \frac{\delta}{3 - 2\delta} V_C
\]

Also, since this is a zero sum game, we know the payoffs will add up to 1:

\[
V_A = 1 - W_B - W_C
\]
\[
V_B = 1 - W_A - W_C
\]
\[
V_C = 1 - W_A - W_B
\]

We can solve this system of equations to find the payoffs.

\[
V_A = 1 - \frac{2\delta}{3 - 2\delta} V_A
\]

\[
\frac{3}{3 - 2\delta} V_A = 1
\]
\[ V_A = 1 - \frac{2}{3} \delta \]

\[ W_A = \frac{\delta}{3 - 2\delta} \left( \frac{3 - 2\delta}{3} \right) = \delta/3 \]

4.2

Strategy: everyone accepts after any history, proposers propose an arbitrary allocation. Any such strategy profile is a NE, since unilateral deviations don’t change the outcome—if all 3 accept regardless of what anyone else does, than one agent rejecting will still leave two accepters, and the outcome will still be to accept the allocation. Thus no profitable deviations.